Chapter 8

POLYNOMIAL EQUATIONS

An nth degree polynomial with complex coefficients is of the form

$$a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$$

where the a_i are complex numbers and $a_0 \neq 0$. (Of course, the coefficients $a_0, \ldots, a_n$ may be real numbers, since a real number is a special case of a complex number.)

Thus a first degree polynomial is of the form $ax + b$ and a second degree polynomial of the form $ax^2 + bx + c$, with $a \neq 0$ in both cases. A non-zero constant a is a polynomial of degree zero; the constant zero is also a polynomial, but it is not assigned a degree.

The polynomial equation $y = x^2 - 6x + 1$ defines y to be a **function of *x*** on the domain of all complex numbers; that is, it provides a rule for assigning a unique complex number y to each complex number x. The table

x	4	3	2	1	0	i	$2i$
$y = x^2 - 6x + 1$	-7	-8	-7	-4	1	$-6i$	$-3 - 12i$

shows that this functional rule assigns -4 to 1, 1 to 0 , $-6i$ to i, etc. A rule that makes y a function of x assigns precisely one value y to a fixed x; however, the same number y may be assigned to more than one x, as is seen here with -7 assigned to 2 and to 4.

It is sometimes convenient to represent the rule that defines y to be a function of x by the symbol $f(x)$. This notation enables one to express in a simple way the number assigned to a given x by the function. For example, $f(1)$, $f(2)$, and $f(3)$ stand for the numbers assigned to 1, 2, and 3, respectively. If $f(x) = x^2 - 6x + 1$, then $f(1) = -4$, $f(2) = -7$, $f(3) = -8$,

$$f(\sqrt{2}) = (\sqrt{2})^2 - 6(\sqrt{2}) + 1,$$

$$f(a + b) = (a + b)^2 - 6(a + b) + 1,$$

and

$$f(x + 1) = (x + 1)^2 - 6(x + 1) + 1.$$

Notice that $f(a + b)$ is not necessarily the same as $f(a) + f(b)$, since $f(a + b)$ is the result of replacing x in $x^2 - 6x + 1$ by $a + b$ and is *not* f times $a + b$.

If several functions are involved in a given discussion, one may use $g(x)$, $F(x)$, $p(x)$, $q(x)$, and so on, as alternates for $f(x)$.

8.1 THE FACTOR AND REMAINDER THEOREMS

If an nth degree polynomial $p(x) = a_0x^n + a_1x^{n-1} + ... + a_n$ can be factored in the form

$$p(x) = a_0(x - r_1)(x - r_2)...(x - r_n), \quad a_0 \neq 0,$$

then the roots of the polynomial equation $p(x) = 0$ are found by setting each of the factors equal to zero, since a product of complex numbers is zero if and only if at least one of the factors is zero. Therefore, the roots are $r_1, ..., r_n$. We wish to establish a form of converse to this result: we wish to show that if r is a root of a polynomial equation $p(x) = 0$ then it follows that $x - r$ is a factor of $p(x)$; that is $p(x)$ can be expressed in the form

$$p(x) = (x - r)q(x)$$

where $q(x)$ is a polynomial in x.

THE FACTOR THEOREM: Let

$$p(x) = a_0x^n + a_1x^{n-1} + ... + a_n$$

be a polynomial in x. If r is a root of $p(x)$, that is, if $p(r) = 0$, then $x - r$ is a factor of $p(x)$.

Proof: Using the hypothesis that $p(r) = 0$, we have

$$\begin{aligned} p(x) &= p(x) - 0 \\ p(x) &= p(x) - p(r) \\ p(x) &= (a_0x^n + a_1x^{n-1} + \cdots + a_n) - (a_0r^n + a_1r^{n-1} + \cdots + a_n) \\ p(x) &= a_0(x^n - r^n) + a_1(x^{n-1} - r^{n-1}) + \cdots + (a_n - a_n). \end{aligned} \tag{1}$$

Since $x - r$ is a factor of $x^n - r^n$, $x^{n-1} - r^{n-1}$, and so on (see Example 2, Chapter 5), it follows that $x - r$ is a factor of the entire right side of equation (1), and so is a factor of $p(x)$.

We next use this theorem to obtain information concerning the case in which r is not a root of $p(x)$.

THE REMAINDER THEOREM: Let $p(x)$ be a polynomial. Then for every complex number r there is a polynomial $q(x)$ such that

$$p(x) = (x - r)q(x) + p(r) \tag{2}$$

Proof: Let us define a new polynomial $f(x)$ by

$$f(x) = p(x) - p(r)$$

Then $f(r) = p(r) - p(r) = 0$. Hence r is a root of $f(x)$ and, by the Factor Theorem, above, $x - r$ is a factor of $f(x)$, and so there is a polynomial $q(x)$ such that

$$f(x) = (x - r)q(x)$$

Now $p(x) - p(r) = (x - r)q(x)$, since both sides are equal to $f(x)$; equation (2) is then obtained by transposing $p(r)$.

The polynomial $p(r)$ is the **remainder** in the division of $p(x)$ by $x - r$. In specific cases, the **quotient** polynomial $q(x)$ of (2), above, may be found by long division or by a more compact form of division called synthetic division. We first illustrate these techniques on the example in which

$$p(x) = x^3 - 7x^2 + 4x + 9, \qquad r = 2.$$

Dividing $p(x)$ by $x - 2$, we have

$$
\begin{array}{r|rrrr}
 & x^2 & -5x & -6 & \\
\cline{2-5}
x - 2 & x^3 & -7x^2 & +4x & +9 \\
 & x^3 & -2x^2 & & \\
\cline{2-4}
 & & -5x^2 & +4x & \\
 & & -5x^2 & +10x & \\
\cline{3-5}
 & & & -6x & +9 \\
 & & & -6x & +12 \\
\cline{4-5}
 & & & & -3
\end{array}
$$

This shows that

$$x^3 - 7x^2 + 4x + 9 = (x - 2)(x^2 - 5x - 6) - 3.$$

That is, $p(x) = (x - 2)q(x) + p(2)$, with $q(x) = x^2 - 5x - 6$ and $p(2) = -3$.

The synthetic form of the division is as follows:

$$
\begin{array}{r|rrrr}
2 & 1 & -7 & 4 & 9 \\
 & & 2 & -10 & -12 \\
\hline
 & 1 & -5 & -6 & \boxed{-3}
\end{array}
$$

The steps in this synthetic form of the division are explained in the treatment of the general case which follows.

The synthetic division of

$$p(x) = a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$$

by $x - h$ is in the form

$$\begin{array}{r|cccccc}
h & a_0 & a_1 & a_2 & \cdots & a_{n-1} & a_n \\
 & & b_1 & b_2 & \cdots & b_{n-1} & b_n \\
\hline
 & c_0 & c_1 & c_2 & \cdots & c_{n-1} & \boxed{c_n}
\end{array}$$

where $c_0 = a_0$, $b_1 = hc_0$, $c_1 = a_1 + b_1$, $b_2 = hc_1$, $c_2 = a_2 + b_2$, ..., $b_n = hc_{n-1}$, $c_n = a_n + b_n$. In general, each b is h times the previous c, $c_0 = a_0$, and each succeeding c is the sum of the a and b above it. The last c, c_n, is the value of $p(h)$, and the other c's are the coefficients of $q(x)$ in the formula $p(x) = (x - h)q(x) + p(h)$; they give us the expression

$$p(x) = (x - h)(c_0x^{n-1} + c_1x^{n-2} + ... + c_{n-2}x + c_{n-1}) + c_n .$$

Example 1. Express $p(x) = x^5 + 25x^2 + 7$ in the form $(x + 3)q(x) + p(-3)$.

Solution: We note that $h = -3$ and that $a_0 = 1$, $a_1 = 0$, $a_2 = 0$, $a_3 = 25$, $a_4 = 0$, and $a_5 = 7$ in this problem. The synthetic division is therefore written

$$\begin{array}{r|cccccc}
-3 & 1 & 0 & 0 & 25 & 0 & 7 \\
 & & -3 & 9 & -27 & 6 & -18 \\
\hline
 & 1 & -3 & 9 & -2 & 6 & \boxed{-11}
\end{array}$$

Hence:

$$x^5 + 25x^2 + 7 = (x + 3)(x^4 - 3x^3 + 9x^2 - 2x + 6) - 11.$$

Example 2. Use synthetic division to show that 5 is a root of $p(x) = 2x^3 - 40x - 50 = 0$, and use this fact to solve the equation.

Solution: We divide $p(x)$ by $x - 5$ with the object of showing that the remainder $p(5)$ is zero. Thus:

$$\begin{array}{r|rrrr} 5 & 2 & 0 & -40 & -50 \\ & & 10 & 50 & 50 \\ \hline & 2 & 10 & 10 & \boxed{0} \end{array}$$

This shows us that $p(x) = (x - 5)(2x^2 + 10x + 10)$. The roots of $p(x) = 0$ are therefore obtained from

$$x - 5 = 0, \quad 2(x^2 + 5x + 5) = 0$$

as 5 and $(-5 \pm \sqrt{25 - 20})/2$; we have, then

$$5, \quad (-5 + \sqrt{5})/2, \quad \text{and} \quad (-5 - \sqrt{5})/2.$$

Example 3. Let $f(x) = 9x^3 + x^2 - 7x + 4$. Find numbers a, b, c, and d such that

$$f(x) = a + b(x + 1) + c(x + 1)^2 + d(x + 1)^3. \tag{3}$$

We give two solutions.

First solution: Letting $x = -1$ in (3), we see that $a = f(-1)$. We therefore use synthetic division to express $f(x)$ in the form $(x + 1)g(x) + f(-1)$ and find that $g(x) = 9x^2 - 8x + 1$ and $a = f(-1) = 3$. Now (3) becomes

$$(x + 1)(9x^2 - 8x + 1) + 3 = 3 + b(x + 1) + c(x + 1)^2 + d(x + 1)^3.$$

On each side we subtract 3 and then divide by $x + 1$, thus obtaining

$$g(x) = 9x^2 - 8x + 1 = b + c(x + 1) + d(x + 1)^2. \tag{4}$$

Letting $x = -1$, we see that b = $g(-1)$. We therefore treat $g(x)$ as $f(x)$ was treated above, and find that $g(x) = (x + 1)(9x - 17) + 18$. Hence $b = 18$. Then (4) becomes

$$(x + 1)(9x - 17) + 18 = b + c(x + 1) + d(x + 1)^2.$$

This leads to

$$9x - 17 = c + d(x + 1)$$

or

$$9(x + 1) - 26 = c + d(x + 1).$$

Hence $c = -26$ and then $d = 9$.

Alternate solution: Let $x + 1 = y$. Then $x = y - 1$ and

$$f(x) = f(y - 1) = 9(y - 1)^3 + (y - 1)^2 - 7(y - 1) + 4.$$

Expanding and collecting like terms, we obtain

$$\begin{aligned} f(x) &= 3 + 18y - 26y^2 + 9y^3 \\ &= 3 + 18(x + 1) - 26(x + 1)^2 + 9(x + 1)^3. \end{aligned}$$

8.2 INTEGRAL ROOTS

Let the coefficients a_i of the polynomial equation

$$a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = 0$$

be integers. Then it can be shown that the only possibilities for integral roots are the integral divisors of the last coefficient a_n. For example, an integer that is a root of

$$x^4 + x^3 + x^2 + 3x - 6 = 0$$

would have to be one of the eight integral divisors ± 1, ± 2, ± 3, ± 6 of -6. Trial of each of these eight integers, as in Example 2 in Section 8.1, would show that 1 and -2 are the only integral roots. The work can be reduced, when one root is found, by substituting the quotient polynomial for the original polynomial in further work. Thus

$$\begin{array}{r|rrrrr} 1 & 1 & 1 & 1 & 3 & -6 \\ & & 1 & 2 & 3 & 6 \\ \hline & 1 & 2 & 3 & 6 & 0 \end{array}$$

shows that $x^4 + x^3 + x^2 + 3x - 6 = (x - 1)(x^3 + 2x^2 + 3x + 6)$. Hence, 1 is a root and the other roots are the roots of the equation $x^3 + 2x^2 + 3x + 6 = 0$.

8.3 RATIONAL ROOTS

We now consider a polynomial equation

$$a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = 0, \quad a_0 \neq 0$$

of degree n with integer coefficients a_i. It can be shown that if there is a rational root p/q, with p and q integers having no common integral divisor greater that 1, then p must be an integral divisor of a_n and q must be an integral divisor of a_0. For example, if the rational number p/q in lowest terms is a root of

$$6x^4 - x^3 - 6x^2 - x - 12 = 0$$

then p must be one of the twelve integral divisors $\pm1, \pm2, \pm3, \pm4, \pm6, \pm12$ of -12 and q one of the integral divisors of 6. Without losing any of the possibilities, we may restrict q to be positive, that is, to be one of the integers 1, 2, 3, 6. The possible rational roots, therefore, are

$$\pm1,\ \pm2,\ \pm3,\ \pm4,\ \pm6,\ \pm12,\ \pm1/2,\ \pm3/2,\ \pm1/3,\ \pm2/3,\ \pm4/3,\ \pm1/6.$$

Trials would show that 3/2 and -4/3 are the only rational roots.

Example. Prove that $\sqrt{2} + \sqrt{3}$ is not a rational number.

Solution: Let $a = \sqrt{2} + \sqrt{3}$. Then

$$\begin{aligned} a - \sqrt{2} &= \sqrt{3} \\ a^2 - 2\sqrt{2}a + 2 &= 3 \\ a^2 - 1 &= 2\sqrt{2}a \\ a^4 - 2a^2 + 1 &= 8a^2 \\ a^4 - 10a^2 + 1 &= 0. \end{aligned}$$

Hence a is a root of $x^4 - 10x^2 + 1 = 0$. This fourth degree polynomial equation has integer coefficients. The rule on rational roots tells us that the only possible rational roots are 1 and -1. Substituting, we see that neither 1 nor -1 is a root. Hence there are no rational roots. Since a is a root, it follows that a is not rational.

Problems for Sections 8.1, 8.2, and 8.3

1. Express $p(x) = x^4 + 5x^3 - 10x - 12$ in the form $(x + 2)q(x) + p(-2)$.

2. Express $f(x) = 5x^5 - x^4 - x^3 - x^2 - x - 2$ in the form $(x - 1)g(x) + f(1)$.

3. Show that -1 is a root of $x^3 + 3x^2 - 2 = 0$, and find the other roots.

4. Show that 2 is a root of $x^3 - 6x + 4 = 0$, and find the other roots.

5. Find a, given that -4 is a root of $5x^6 - 7x^5 + 11x + a = 0$.

6. Find b, given that 3 is a root of $x^7 - 10x^5 + 8x^3 + 4x^2 - 3x + b = 0$.

7. Find all the integral roots of $x^4 - 2x^3 - x^2 - 4x - 6 = 0$, and then find the other roots.

8. Find all the integral roots of $x^5 - 8x^4 + 15x^3 + 8x^2 - 64x + 120 = 0$, and then find the other roots.

9. Let $f(x) = (x - a)^3 - x^3 + a^3$. Find $f(0)$ and $f(a)$, and use this information to find two factors of $f(x)$.

10. Let $g(x) = (x - a)^5 - x^5 + a^5$. Show that $f(x)$ is divisible by x and by $x - a$, and find the other factors.

11. Find all the integral roots of $3x^4 + 20x^3 + 36x^2 + 16x = 0$, and then find the other roots.

12. Let $f(x) = a_0x^n + a_1x^{n-1} + ... + a_{n-1}x$; that is, let $a_n = 0$. Also, let the a_i be integers. Show that any non-zero integral root of $f(x) = 0$ is an integral divisor of a_{n-1}.

13. Find a rational root of $3x^3 + 4x^2 - 21x + 10 = 0$, and then find the other roots.

14. Find all the roots of $6x^4 + 31x^3 + 25x^2 - 33x + 7 = 0$.

15. Find all the roots of $81x^5 - 54x^4 + 3x^2 - 2x = 0$.

16. Let $f(x) = a_0x^n + a_1x^{n-1} + ... + a_{n-1}x$ with the a_i integers. State a necessary condition for a non-zero rational number to be a root of $f(x) = 0$.

17. Given that a and b are integers, what are possibilities for rational roots of $x^3 + ax^2 + bx + 30 = 0$?

18. Let $f(x) = x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$ with the a_i integers. Note that $a_0 = 1$. Show that any rational root of $f(x) = 0$ must be an integer.

19. Let $f(x)$ be a polynomial. Let r and s be roots of $f(x) = 0$ and let $r \neq s$. Show that there exist polynomials $g(x)$ and $h(x)$ such that all of the following are true:

 (a) $f(x) = (x - r)g(x)$.
 (b) $g(s) = 0$.
 (c) $g(x) = (x - s)h(x)$.
 (d) $f(x) = (x - r)(x - s)h(x)$.

20. Let $f(x) = 0$ be a polynomial equation with distinct roots r, s, and t. Show that $f(x) = (x - r)(x - s)(x - t)p(x)$, with $p(x)$ a polynomial.

21. Prove that if $r_1, r_2 \ldots, r_n$ are distinct roots of a polynomial equation $f(x) = 0$, then $f(x)$ is a multiple of $(x - r_1)(x - r_2)\ldots(x - r_n)$.

22. Prove that $\sqrt{3} - \sqrt{2}$, $\sqrt{2} - \sqrt{3}$, and $-\sqrt{2} - \sqrt{3}$ are all irrational.

23. Prove that $\sqrt{5} + \sqrt{3}$, $\sqrt{5} - \sqrt{3}$, $-\sqrt{5} + \sqrt{3}$, and $-\sqrt{5} - \sqrt{3}$ are all irrational.

24. Prove that $\sqrt[3]{14}$ is irrational.

25. Find an eighth-degree polynomial equation with integer coefficients that has

$\sqrt{2} + \sqrt{3} + \sqrt{7}$ as a root.

26. If $f(x)$ is a function of x, the notation $\Delta f(x)$ represents $f(x + 1) - f(x)$. Show that

$\Delta x^2 = 2x + 1$ and $\Delta x^3 = 3x^2 + 3x + 1$.

27. Let $\Delta f(x) = f(x + 1) - f(x)$. Find $\Delta f(x)$ for each of the following:

(a) $f(x) = a + bx$.
(b) $f(x) = a + bx + cx^2$.
(c) $f(x) = a + bx + cx^2 + dx^3$.
(d) $f(x) = x^n$, with n a positive integer.

28. Find $f(x + 2) - 2f(x + 1) + f(x)$ for:

(a) $f(x) = a + bx$.
(b) $f(x) = a + bx + cx^2$.

29. Find $f(x + 3) - 3f(x + 2) + 3f(x + 1) - f(x)$ for:

(a) $f(x) = a + bx$.
(b) $f(x) = a + bx + cx^2$.
(c) $f(x) = a + bx + cx^2 + dx^3$.

30. Let $\Delta^n f(x)$, with n a positive integer, be defined inductively by

$$\begin{aligned}
\Delta^1 f(x) &= \Delta f(x) = f(x+1) - f(x),\\
\Delta^2 f(x) &= \Delta[\Delta f(x)] = \Delta[f(x+1) - f(x)]\\
&= [f(x+2) - f(x+1)] - [f(x+1) - f(x)],\\
\Delta^3 f(x) &= \Delta[\Delta^2 f(x)],\\
&\ldots,\\
\Delta^{m+1} f(x) &= \Delta[\Delta^m f(x)],\\
&\ldots .
\end{aligned}$$

[The function $\Delta^n f(x)$ is called the nth difference of $f(x)$.] Show that

$$\Delta^n f(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x+n-k).$$

31. Let $\Delta^n f(x)$ be defined as in Problem 30 above and show:

(a) $\Delta^n f(x) = 0$, if $f(x)$ is a polynomial of degree less than n.

(b) $\Delta^n f(x) = n! a_0$, if $f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n$.

32. Let $f(x) = a + bx + cx^2$. Let $r = f(0)$, $s = f(1) - f(0)$, and $t = f(2) - 2f(1) + f(0)$.

(a) Show that $f(x) = r + sx + tx(x - 1)/2$.
(b) Generalize this problem.

33. Let $f(x) = 5x^4 - 6x^3 - 3x^2 + 8x + 2$. Use repeated synthetic division to find numbers a, b, c, d, and e such that

$$f(x) = a + b(x-2) + c(x-2)^2 + d(x-2)^3 + e(x-2)^4.$$

34. Use the method of the alternate solution for Example 3 in Section 8.1 to do Problem 33.

35. Let $f(x) = x^3 + ax^2 + bx + c$, and let a, b, c, and r be complex numbers. Show that

$$f(x) = f(r) + (3r^2 + 2ar + b)(x - r) + s(x - r)^2 + (x - r)^3,$$

and express s in terms of a and r.

36. Let $f(x) = x^3 + ax^2 + bx + c$, and let r be a root of $f(x) = 0$. Show that $f(x)$ is divisible by $(x - r)^2$ if and only if $3r^2 + 2ar + b = 0$.

37. Let $f(x) = x^3 + ax^2 + bx + c$, $g(x) = 3x^2 + 2ax + b$, and $h(x) = 6x + 2a$. Show that $f(x) = (x - r)^3$ if and only if $f(r) = g(r) = h(r) = 0$.

38. Let $f(x) = x^4 + ax^3 + bx^2 + cx + d$. Find s, t, and u in terms of a, b, c, and r such that

$$f(x) = f(r) + s(x - r) + t(x - r)^2 + u(x - r)^3 + (x - r)^4.$$

39. Do the methods of this chapter enable you to solve $x^3 - 3x + 1 = 0$?

8.4 SYMMETRIC FUNCTIONS

If we multiply out $(x - a)(x - b)(x - c)(x - d)$, we obtain an expression of the form $x^4 - s_1x^3 + s_2x^2 - s_3x + s_4$, where

$$s_1 = a + b + c + d,$$
$$s_2 = ab + ac + ad + bc + bd + cd,$$
$$s_3 = abc + abd + acd + bcd,$$
$$s_4 = abcd.$$

We note that s_k is the sum of all products of a, b, c, and d taken k at a time. It is also clear that the s_k are **symmetric functions** of a, b, c, and d; that is, they do not change value when any two of a, b, c, and d are interchanged.

The **Fundamental Theorem of Algebra**, the proof of which is too advanced for this book, states that the general nth degree polynomial with complex coefficients

$$a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n, \quad a_0 \neq 0$$

has a factorization into linear factors

$$a_0(x - r_1)(x - r_2) \ldots (x - r_n)$$

where the r_i are complex numbers. One can then see that $(-1)^k a_k/a_0$ is the sum of all the products of k factors chosen from $r_1, r_2, \ldots, r_n$.

The **absolute value** of a real number x is written as $|x|$ and is defined as follows: If $x \geq 0$, then $|x| = x$; if $x < 0$, then $|x| = -x$.

Problems for Section 8.4

1. Let $3(x - r)(x - s) = 3x^2 - 12x + 8$. Find the following:

(a) $r + s$.
(b) rs.
(c) $(r + s)^2$.
(d) $r^2 + s^2$.
(e) $r^2 - 2rs + s^2$.
(f) $|r - s|$.

2. Find the sum, product, and absolute value of the difference of the roots of $5x^2 + 7x - 4 = 0$.

3. Let $(x - r)(x - s) = x^2 + x + 1$. Show the following:

   (a) $r = -(s + 1)$, $s = -(r + 1)$.
   (b) $r^3 + r^2 + r = 0 = s^3 + s^2 + s$.
   (c) $r = s^2$, $s = r^2$.
   (d) $r^{-1} + s^2 = -1$, $s^{-1} + r^2 = -1$.
   (e) $r^4 + r^{-1}s^{-1} + s^4 = 0$.
   (f) $r^9 - r^6 + r^3 - 1 = 0 = s^9 - s^6 + s^3 - 1$.
   (g) $r^{10} + s^7 + r^4 + s = -2 = s^{10} + r^7 + s^4 + r$.
   (h) $(r^2 - r + 1)(s^2 - s + 1)(r^4 - r^2 + 1)(s^4 - s^2 + 1) = 16$.

4. Let r be a root of $x^2 + x + 1 = 0$. Show the following:

   (a) $x^3 - a^3 = (x - a)(x - ar)(x - ar^2)$.
   (b) $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + ry + r^2z)(x + r^2y + \mathrm{r}\mathrm{z})$.

5. Let a, b, and c be the roots of $x^3 + 3x + 3 = 0$. Find $(a + 1)(b + 1)(c + 1)$.

6. Given that $(x - a)(x - b) = x^2 - px + q$, express each of the following in terms of p and q:

   (a) $a + b$.
   (b) ab.
   (c) $a^2 + 2ab + b^2$.
   (d) $a^2 + ab + b^2$.
   (e) $ab(a^2 + ab + b^2)$.
   (f) a^3b^3.
   (g) The coefficients of the expansion of $(x - a^2)(x - ab)(x - b^2)$.

7. Let $(x - r)(x - s) = x^2 - px + q$, and let $(x - r^3)(x - r^2s)(x - rs^2)(x - s^3) = x^4 - ax^3 + bx^2 - cx + d$. Express a, b, c, and d in terms of p and q.

8. Let $(x - a)(x - b) = x^2 - ex + f$, $(x - c)(x - d) = x^2 - gx + h$, and $(x - ac)(x - ad)(x - bc)(x - bd) = x^4 - px^3 + qx^2 - rx + s$. Find p, q, r, and s in terms of e, f, g, and h.

9. Let $(x - a)(x - b)(x - c) = x^3 - 3x + 1$. Find each of the following:

   (a) $2(a + b + c)$.
   (b) $(a + b)(a + c) + (a + b)(b + c) + (a + c)(b + c)$.
   (c) $(a + b)(a + c)(b + c)$.
   (d) the equation $y^3 - py^2 + qy - r = 0$ whose roots are $a + b$, $a + c$, and $b + c$.

10. Do Problem 9 with $(x - a)(x - b)(x - c) = x^3 + 3x - 1$.

11. Let $s_1 = a + b + c$, and $s_2 = ab + ac + bc$, and $s_3 = abc$. Find numbers x, y, z, t, u, v, and w such that for all a, b, and c:

(a) $a^3 + b^3 + c^3 = xs_3 + ys_1s_2 + zs_1^3$.
(b) $a^4 + b^4 + c^4 = ts_1s_3 + us_2^2 + vs_1^2s_2 + ws_1^4$.

12. Let $(x - r)(x - s)(x - t) = x^3 - ax^2 + bx - c$. Express $r^5 + s^5 + t^5$ in terms of a, b, and c.

13. Let $(x -1)(x -2)(x - 3)...(x - n) = x^n - s_1x^{n-1} + s_2x^{n-2} - ... + (-1)^n s_n$. Show the following:

(a) $s_1 = \binom{n+1}{2}$.

(b) $s_n = n!$.

(c) $2s_2 = (1^3 + 2^3 + ... + n^3) - (1^2 + 2^2 + ... + n^2)$.